Modelowanie preferencji a ryzyko'21

# Pareto Efficiency of Mixed Quantum Strategy Equilibria in Frąckiewicz-Pykacz parametrization of EWL model

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# Games and probability distributions

We consider two player games

$$G = \left(N, \{S_X\}_{X \epsilon N}, \{P_X\}_{X \epsilon N}\right)$$

where:

$N = \{A, B\}$ is the set of players

$S_A = \{A_0, A_1\}, S_B = \{B_0, B_1\}$ are possible pure strategies

$P_X: S_A \times S_B \rightarrow \{v_{ij}^X \epsilon \mathbb{R} \,|\, i, j = 0,1\}, X = A, B$, are payoff functions, represented by the game bimatrix

$$\begin{pmatrix} (v_{00}^A, v_{00}^B) & (v_{01}^A, v_{01}^B) \\ (v_{10}^A, v_{10}^B) & (v_{11}^A, v_{11}^B) \end{pmatrix}$$

Let

$$\Delta(S_A \times S_B) = \left\{\sum_{i,j=0,1} \sigma_{ij} A_i B_j \,\middle|\, \sigma_{ij} \geq 0, \sum_{i,j=0,1} \sigma_{ij} = 1\right\}$$

be the set of *probability distributions* over $S_A \times S_B$

# Mixed strategies and Nash equilibria

If the set of probability distributions can be factorized

$$\begin{pmatrix} \sigma_{00} & \sigma_{01} \\ \sigma_{10} & \sigma_{11} \end{pmatrix} = \begin{pmatrix} \sigma_A \sigma_B & \sigma_A(1-\sigma_B) \\ (1-\sigma_A)\sigma_B & (1-\sigma_A)(1-\sigma_B) \end{pmatrix}$$

they define mixed strategies $\sigma_A, \sigma_B \epsilon\ [0,1]$.

The *mixed classical game* is

$$G^{mix} = (N, \Delta S_A, \Delta S_B, \Delta P_A, \Delta P_B)$$

where $\Delta(S_X) = \{\sigma_X X_0 + (1-\sigma_X)X_1 \mid 0 \leq \sigma_X \leq 1\} \equiv [0,1]$.

Mixed strategies form a subset of all probability distributions

$$\Delta S_A \times \Delta S_B \subset \Delta(S_A \times S_B)$$

The pair of strategies $(\sigma_A^*, \sigma_B^*)\ \epsilon\ \Delta S_A \times \Delta S_B$ is a Nash equilibrium iff

$$\Delta P_A(\sigma_A^*, \sigma_B^*) \geq \Delta P_A(\sigma_A, \sigma_B^*) \text{ and } \Delta P_B(\sigma_A^*, \sigma_B^*) \geq \Delta P_B(\sigma_A^*, \sigma_B),$$

for each $\sigma_X\ \epsilon\ \Delta S_X$, $X = A, B$

# Pareto optimality and correlated equilibria

A pair of strategies $(\sigma_A, \sigma_B) \epsilon S$ is not Pareto optimal in $S$ if there exists another pair $(\sigma_A{}', \sigma_B{}') \epsilon S$ that is better for one of the players and not worse for the other. Otherwise $(\sigma_A, \sigma_B) \epsilon S$ is called *Pareto optimal*.

Probability distribution $\{\sigma_{ij}\}_{i,j=0,1}$ over set of strategies $(A_i, B_j)_{i,j=0,1}$ of the game $G$ is a *correlated equilibrium* iff

$$\sum_{j=0,1} \sigma_{ij}\, v_{ij}^A \geq \sum_{j=0,1} \sigma_{ij}\, v_{-ij}^A \quad \text{and} \quad \sum_{j=0,1} \sigma_{ji}\, v_{ji}^B \geq \sum_{j=0,1} \sigma_{ji}\, v_{j(-i)}^B$$

where $-i \neq i$ is the index if the remaining strategy.

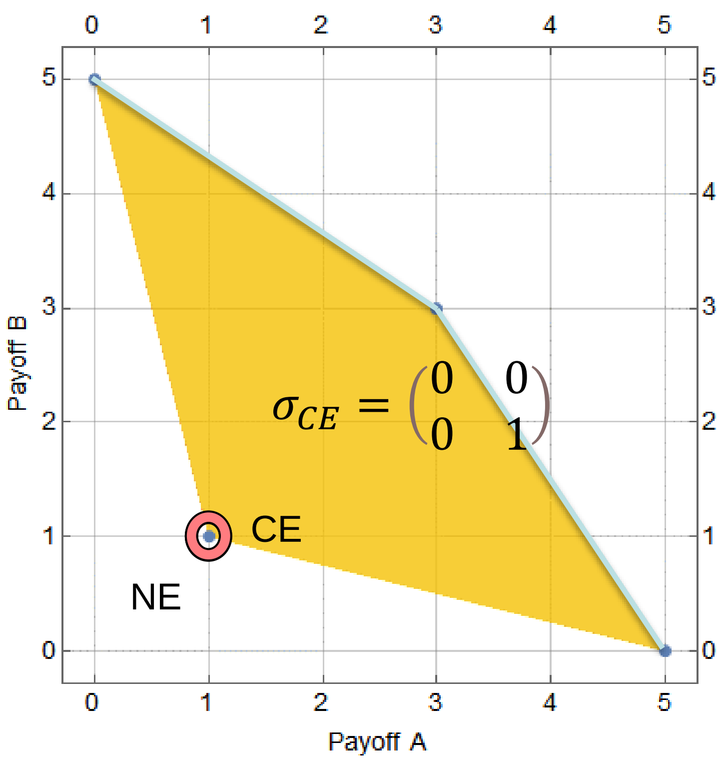
# Efficiency of selected classical games

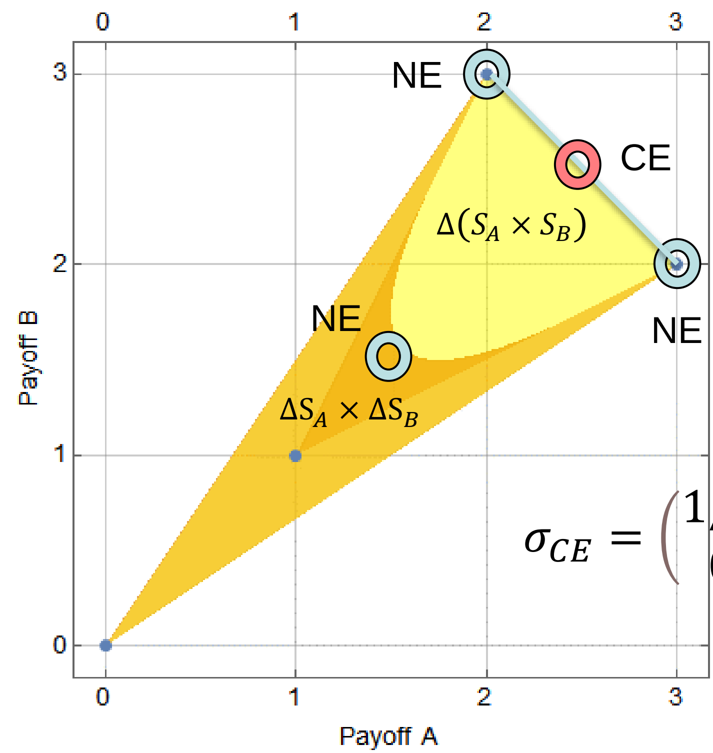
| prisoner's dilemma | | Bob | |
|---|---|---|---|
| | | $B_0$ | $B_1$ |
| Alice | $A_0$ | $(3,3)$ | $(0,5)$ |
| | $A_1$ | $(5,0)$ | $(1,1)$ |

| battle of the sexes | | Bob | |
|---|---|---|---|
| | | $B_0$ | $B_1$ |
| Alice | $A_0$ | $(3,2)$ | $(1,1)$ |
| | $A_1$ | $(0,0)$ | $(2,3)$ |

$$\sigma_{00} = \sigma_{01} = \sigma_{10} = 0$$
$$\sigma_{11} = 1$$

$$3\sigma_{00} \geq \sigma_{01}, \sigma_{00} \geq 3\sigma_{10}$$
$$3\sigma_{11} \geq \sigma_{01}, \sigma_{11} \geq 3\sigma_{10}$$




$$\sigma_{CE} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

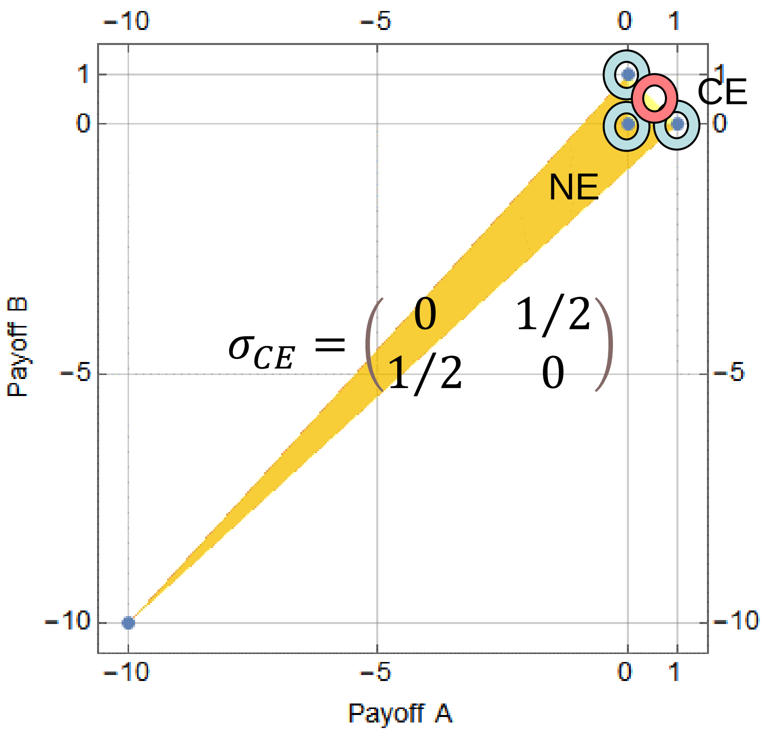
# Efficiency of selected classical games

| chicken | | Driver B | |
|---|---|---|---|
| Driver A | | $B_0$ | $B_1$ |
| | $A_0$ | $(0,0)$ | $(0,1)$ |
| | $A_1$ | $(1,0)$ | $(-10,-10)$ |

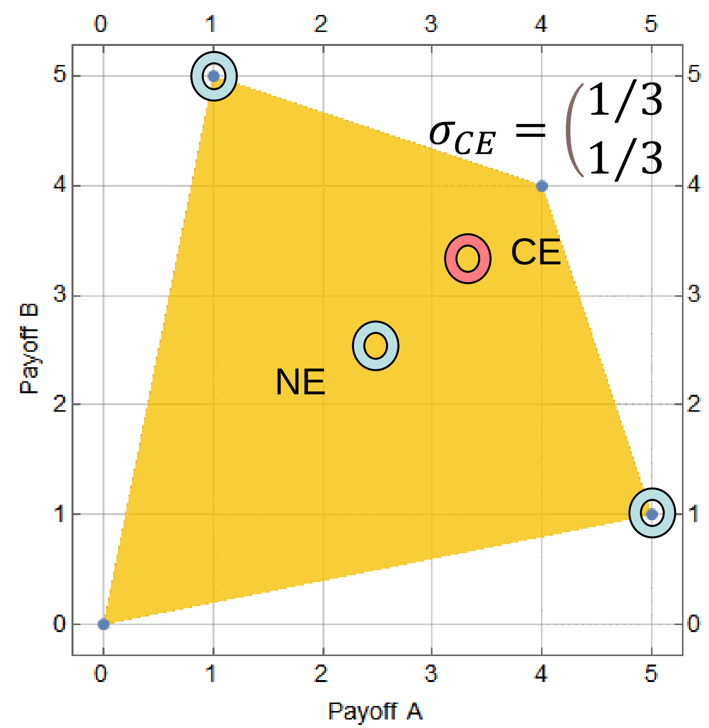
| chicken 2 | | Player B | |
|---|---|---|---|
| Player A | | $B_0$ | $B_1$ |
| | $A_0$ | $(4,4)$ | $(1,5)$ |
| | $A_1$ | $(5,1)$ | $(0,0)$ |

$$\sigma_{00} \le 10\sigma_{01}, \sigma_{00} \le 10\sigma_{10}$$
$$10\sigma_{11} \le \sigma_{01}, 10\sigma_{11} \le \sigma_{10}$$

$$\sigma_{00} \le \sigma_{01}, \sigma_{00} \le \sigma_{10}$$
$$\sigma_{11} \le \sigma_{01}, \sigma_{11} \le \sigma_{10}$$




$$\sigma_{CE} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$




$$\sigma_{CE} = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix}$$

# Quantum games

The quantum game in Eisert-Wilkens-Lewenstein quantization scheme is: (Eisert et al, PRL 83, 3077 (1999)

$$\Gamma_{EWL} = \left(N, \{U_X\}_{X \epsilon N}, \{\Pi_X\}_{X \epsilon N}\right)$$

where:

$N = \{A, B\}$ is the set of players

$\hat{U}_A = \hat{U}(\theta_A, \alpha_A, \beta_A)$, $\hat{U}_B = \hat{U}(\theta_B, \alpha_B, \beta_B)$, are unitary transformations (quantum strategies)

$\Pi_X: SU(2) \times SU(2) \to \mathbb{R}$ are payoff functions:

$$\Pi_X\left(\hat{U}_A, \hat{U}_B\right) = \sum_{i,j=0,1} |p_{ij}|^2 \, v_{ij}^X, \quad X = A, B,$$ where

$$|p_{00}|^2 = \cos\frac{\theta_A}{2}\cos\frac{\theta_B}{2}\cos(\alpha_A + \alpha_B) + \sin\frac{\theta_A}{2}\sin\frac{\theta_B}{2}\sin(\beta_A + \beta_B),$$

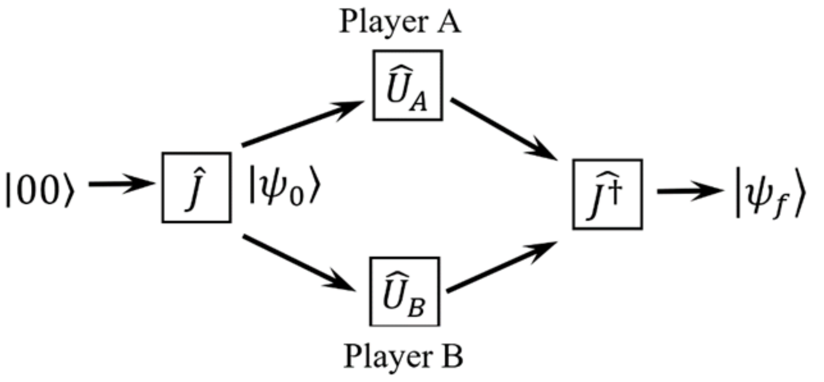
$$|p_{01}|^2 = \cos\frac{\theta_A}{2}\sin\frac{\theta_B}{2}\cos(\alpha_A - \beta_B) + \sin\frac{\theta_A}{2}\cos\frac{\theta_B}{2}\sin(\alpha_B - \beta_A),$$

$$|p_{10}|^2 = \cos\frac{\theta_A}{2}\sin\frac{\theta_B}{2}\sin(\alpha_A - \beta_B) + \sin\frac{\theta_A}{2}\cos\frac{\theta_B}{2}\cos(\alpha_B - \beta_A),$$

$$|p_{11}|^2 = \cos\frac{\theta_A}{2}\cos\frac{\theta_B}{2}\sin(\alpha_A + \alpha_B) - \sin\frac{\theta_A}{2}\sin\frac{\theta_B}{2}\cos(\beta_A + \beta_B).$$

# EWL approach

The quantum EWL approach to the game is




where: $|00\rangle$ is the initial state

$\hat{J} = \frac{1}{\sqrt{2}}(\hat{I} + i\sigma_x \otimes \sigma_x), J^\dagger$ are the entangling, disentangling operators,

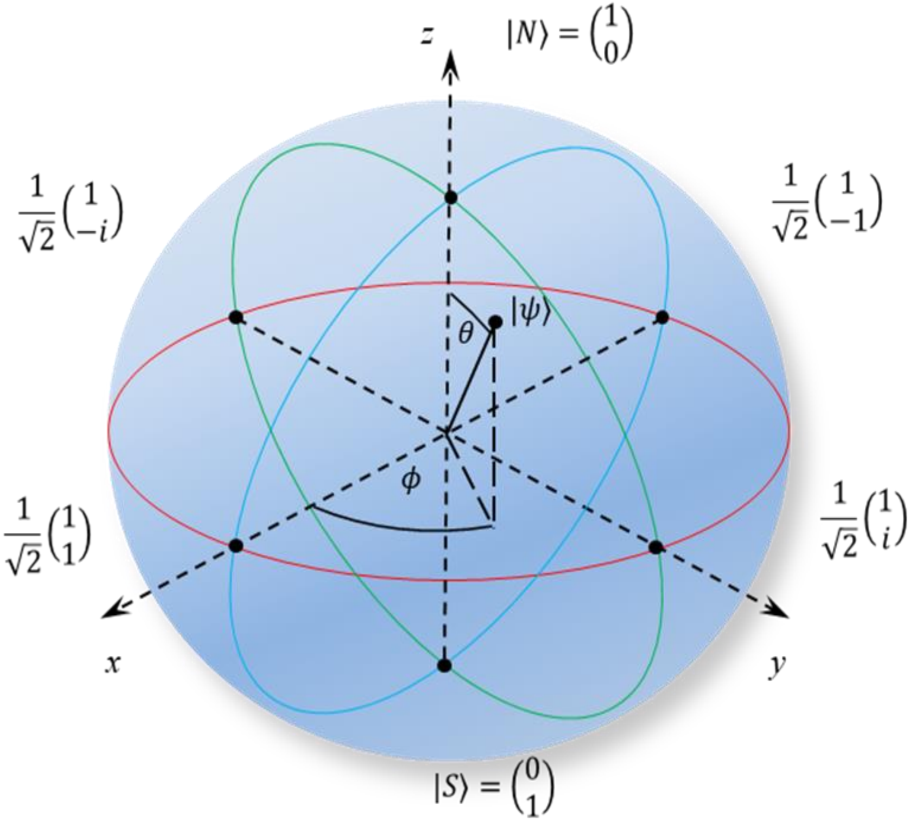
$$\hat{U}_X(\theta_X, \alpha_X, \beta_X) = \begin{pmatrix} e^{i\alpha_X}\cos\frac{\theta_X}{2} & ie^{i\beta_X}\sin\frac{\theta_X}{2} \\ ie^{-i\beta_X}\sin\frac{\theta_X}{2} & e^{-i\alpha_X}\cos\frac{\theta_X}{2} \end{pmatrix}, X = A, B,$$

$\left|\psi_f\right\rangle = \sum_{i,j=0,1} p_{ij}\ |ij\rangle$, is the final state defining the game payoffs

# Quantum strategies

Strategies $\widehat{U}_A = \widehat{U}(\theta_A, \alpha_A, \beta_A)$ and $\widehat{U}_B = \widehat{U}(\theta_B, \alpha_B, \beta_B)$,

$$\widehat{U}_X(\theta_X, \alpha_X, \beta_X) = \begin{pmatrix} e^{i\alpha_X}\cos\frac{\theta_X}{2} & ie^{i\beta_X}\sin\frac{\theta_X}{2} \\ ie^{-i\beta_X}\sin\frac{\theta_X}{2} & e^{-i\alpha_X}\cos\frac{\theta_X}{2} \end{pmatrix},$$ are generated by Pauli strategies:




$$\widehat{P_0} = \widehat{U}(0,0,\beta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\widehat{P_x} = \widehat{U}(\pi,\alpha,\pi) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

$$\widehat{P_y} = \widehat{U}(\pi,\alpha,\pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\widehat{P_z} = \widehat{U}(0,\pi/2,\beta) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

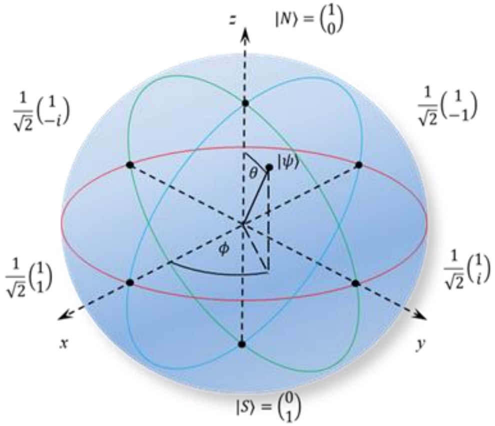
where

$$\widehat{\sigma_x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \widehat{\sigma_y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \widehat{\sigma_z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

are Pauli matrices

# Classical limit of the quantum game

Le us assume $\alpha = \beta = 0$, in this case




$$\hat{U}(\theta, 0, 0) = \cos\frac{\theta}{2}\hat{I} + i\sin\frac{\theta}{2}\sigma_x$$

is equivalent to the classical mixed strategy

| | | B | |
|---|---|---|---|
| A | | $cos^2\frac{\theta_B}{2}$ | $sin^2\frac{\theta_B}{2}$ |
| | $cos^2\frac{\theta_A}{2}$ | $(a_{00}, b_{00})$ | $(a_{01}, b_{01})$ |
| | $sin^2\frac{\theta_A}{2}$ | $(a_{10}, b_{10})$ | $(a_{11}, b_{11})$ |

$\theta_B = 0$ $\theta_B = \pi$

and the payoff is

$$\$_{A(\boldsymbol{B})} = a(b)_{00}\cos^2\frac{\theta_A}{2}\cos^2\frac{\theta_B}{2} + a(b)_{01}\cos^2\frac{\theta_A}{2}\sin^2\frac{\theta_B}{2}$$
$$+a(b)_{10}\sin^2\frac{\theta_A}{2}\cos^2\frac{\theta_B}{2} + a(b)_{11}\sin^2\frac{\theta_A}{2}\sin^2\frac{\theta_B}{2}$$

# EWL with Frąckiewicz-Pykacz parameterization

Let us restrict the set of quantum strategies to

$$\hat{U}_X(\theta_X, \phi_X) = \begin{pmatrix} e^{-i\phi_X} \cos\frac{\theta_X}{2} & -e^{-i\phi_X} \sin\frac{\theta_X}{2} \\ e^{i\phi_X} \sin\frac{\theta_X}{2} & e^{i\phi_X} \cos\frac{\theta_X}{2} \end{pmatrix}$$

$$\widehat{P_0} = \hat{U}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\widehat{P_x} = \hat{U}\left(\pi, \frac{3\pi}{2}\right) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

$$\widehat{P_y} = \hat{U}(\pi, 0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\widehat{P_z} = \hat{U}\left(0, \frac{3\pi}{2}\right) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

- In this parameterization, Nash equilibria in pure strategies are possible
- F-P parametrization is invariant with respect to strongly isomorphic transformation of input games

# Quantum game in Pauli strategies

The payoff matrix of Pauli strategies in the EWL scheme

| | | Player B | | | |
|---|---|---|---|---|---|
| | | $\widehat{P_0}$ | $\widehat{P_x}$ | $\widehat{P_y}$ | $\widehat{P_z}$ |
| Player A | $\widehat{P_0}$ | $(a_{00}, b_{00})$ | $(a_{01}, b_{01})$ | $(a_{10}, b_{10})$ | $(a_{11}, b_{11})$ |
| | $\widehat{P_x}$ | $(a_{10}, b_{10})$ | $(a_{11}, b_{11})$ | $(a_{00}, b_{00})$ | $(a_{01}, b_{01})$ |
| | $\widehat{P_y}$ | $(a_{01}, b_{01})$ | $(a_{00}, b_{00})$ | $(a_{11}, b_{11})$ | $(a_{10}, b_{10})$ |
| | $\widehat{P_z}$ | $(a_{11}, b_{11})$ | $(a_{10}, b_{10})$ | $(a_{01}, b_{01})$ | $(a_{00}, b_{00})$ |

one can construct mixed Pauli strategies defined by quadruples of coefficients:

$$\Delta U_X \equiv \Delta(U_X) = \{\sum_{\alpha=0,x,y,z} \sigma_\alpha^X \widehat{P_\alpha} \,\big|\, 0 \leq \sigma_\alpha^X;\ \sum_{\alpha=0,x,y,z} \sigma_\alpha^X = 1\},\ X = A, B,$$

# Quantum Mixed Equilibria

| prisoner's dilemma | | Bob | |
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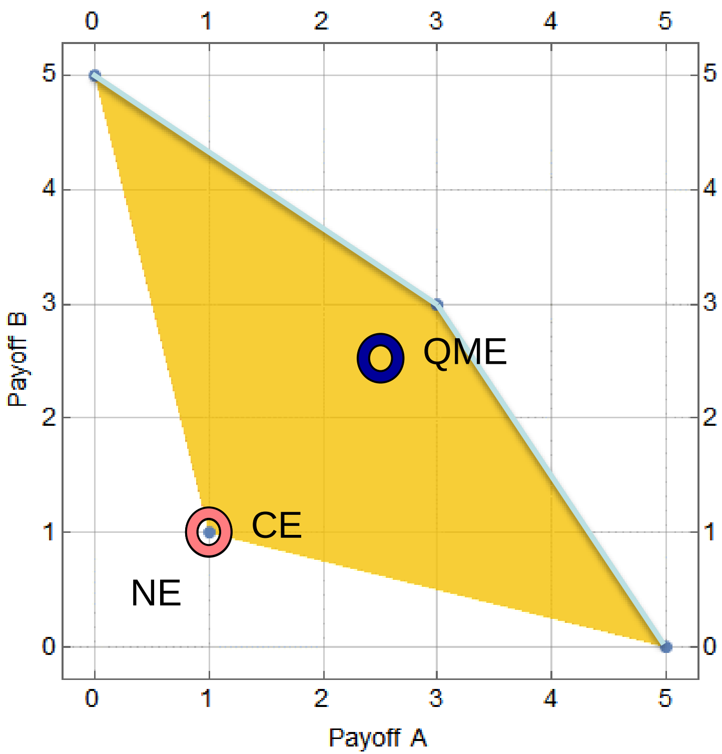
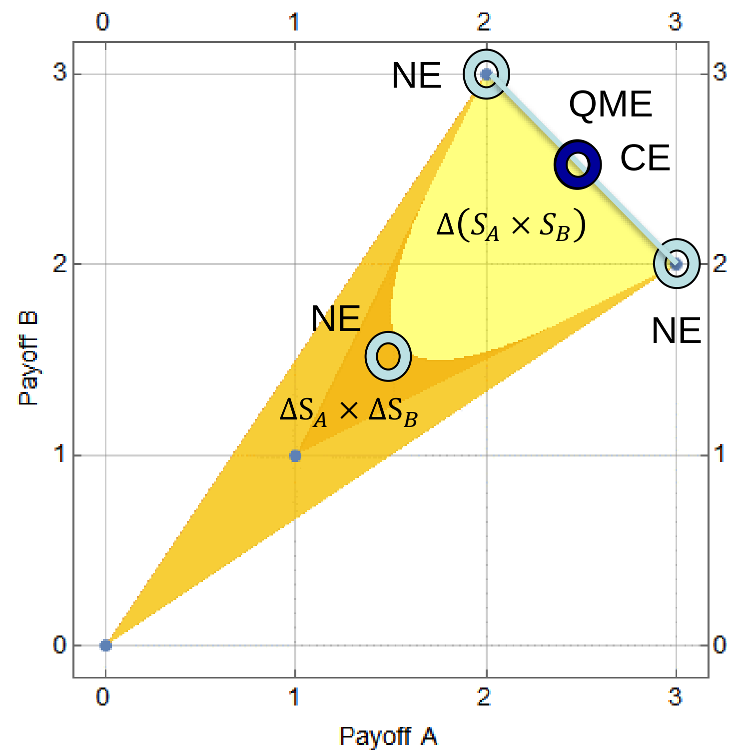
| | $A_1$ | $(0,0)$ | $(2,3)$ |

$$\sigma^A = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right), \sigma^B = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$$




$$\sigma^A = \sigma^B = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$$



# Quantum Mixed Equilibria

| chicken | | Driver B | |
|---|---|---|---|
| Driver A | | $B_0$ | $B_1$ |
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| chicken 2 | | Player B | |
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| Player A | | $B_0$ | $B_1$ |
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$$\sigma^A = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right), \sigma^B = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$$

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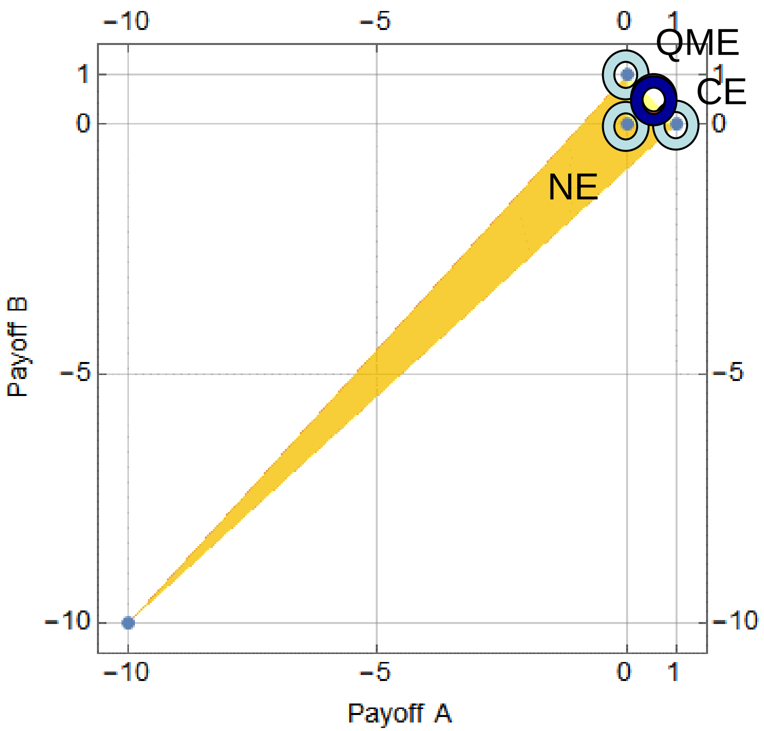
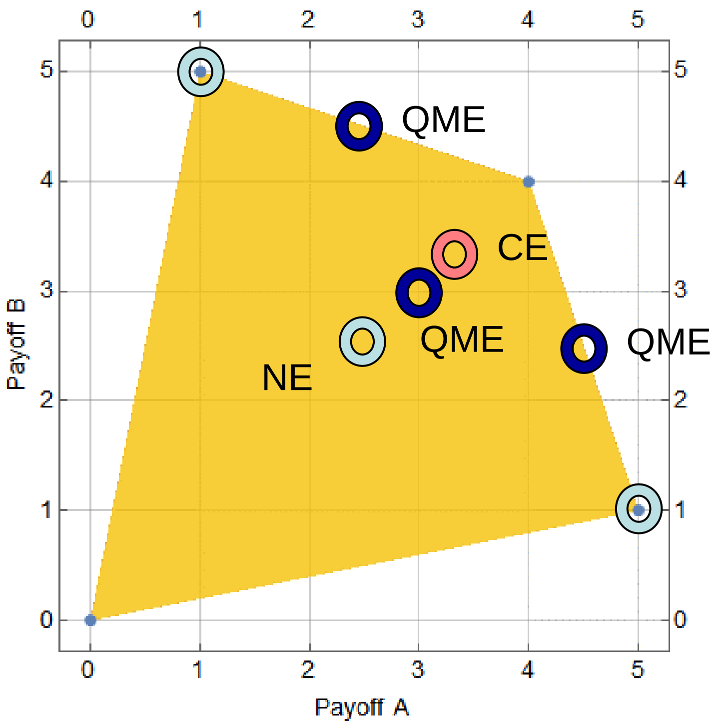
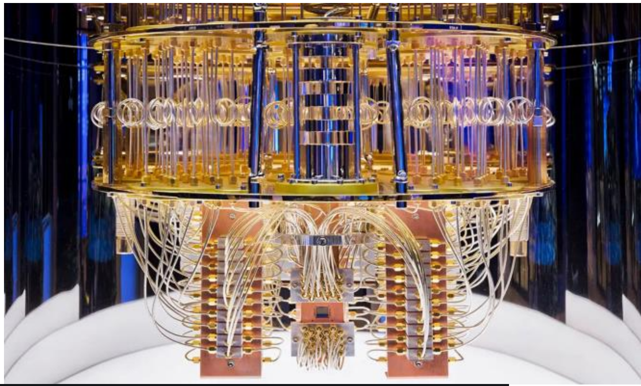
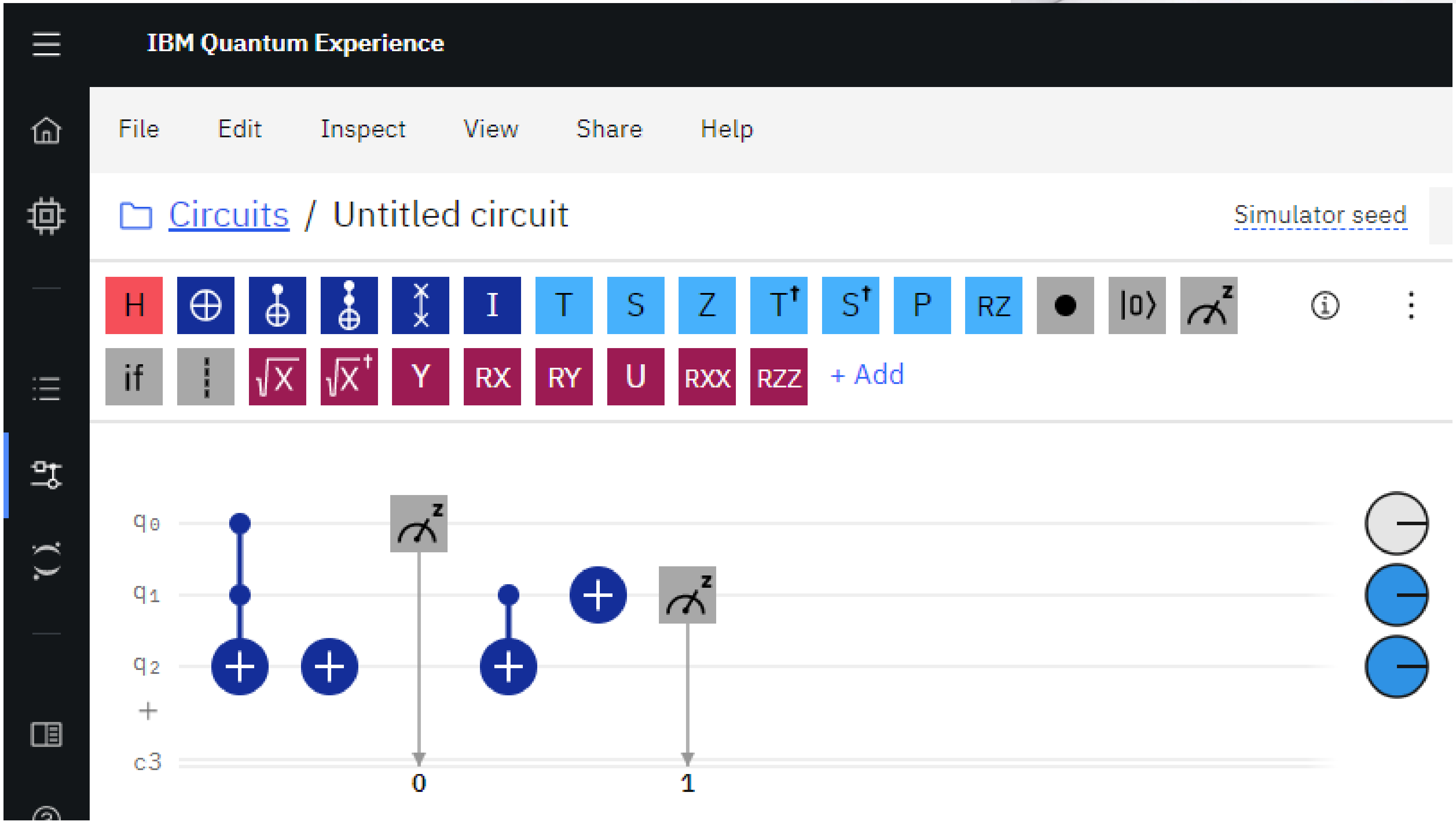


# Quantum Computer

https://quantum-computing.ibm.com/





# Conclusions

1. In some games the set of mixed strategies is a proper subset of the set of probability distributions
2. Correlated equilibria significantly improve paretoefficiency of Nash equilibria
3. Quantum games give players new strategies not available in classic games and strongly depend on the parameterization used
4. Nash equilibria of quantum in mixed strategies are close to paretoefficiency of correlated equilibria
5. FP parameterization provides a strong isomorphism of the quantum game and gives the same Nash equilibria in mixed strategies as full SU(2) parameterization of EWL